

PREORDERINGS COMPATIBLE WITH PROBABILITY MEASURES¹

BY

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ABSTRACT. The main theorem proved in this paper is:

Let B be a σ -complete Boolean algebra and \succcurlyeq a binary relation with field B such that:

(i) Every finite subalgebra B' admits a probability measure μ' such that for $p, q \in B'$, $p \succcurlyeq q$ iff $\mu' p \geq \mu' q$.

(ii) If for every i , $p_i, q \in B$ and $p_i \subseteq p_{i+1} \leq q$, then $\bigcup_{i < \infty} p_i \leq q$.

Under these conditions there is a σ -additive probability measure μ on B such that:

(a) If there is a $p \in B$, such that for every $q \subseteq p$ there is a $q' \subseteq q$ with $q' \leq q$, $q' \not\leq 0$, and $q \not\leq q'$, then we have that for every $p, q \in B$, $\mu p \geq \mu q$ iff $p \succcurlyeq q$.

(b) If for every $p \in B$, there is a $q \subseteq p$ such that $q' \subseteq q$ implies $q \leq q'$ or $q' \leq 0$, then we have that for every $p, q \in B$, $p \succcurlyeq q$ implies $\mu p \geq \mu q$.

1. The main purpose of this work is to find necessary and sufficient conditions for the existence of σ -additive probability measures compatible with preorderings (i.e. weak orderings).

These types of considerations are motivated by philosophical, heuristic, and mathematical problems on the foundations of probability. For instance, let \mathfrak{B} be the Boolean algebra of events. It is often not feasible, or even possible, to directly assign probabilities to the events. However it may be possible to qualitatively compare events with respect to probability. Thus, we may be able to introduce a preordering \succcurlyeq on \mathfrak{B} which can be intuitively interpreted as meaning "qualitatively at least as possible as". In this form the natural question arises about conditions on \mathfrak{B} and \succcurlyeq for the existence of a probability measure μ on \mathfrak{B} such that for all $p, q \in \mathfrak{B}$, we have

$$p \succcurlyeq q \text{ iff } \mu p \geq \mu q.$$

(In this case we say that μ is compatible with \succcurlyeq .) The least we could ask is that

$$p \succcurlyeq q \text{ imply } \mu p \geq \mu q.$$

(For this case we say that μ is weakly compatible with \succcurlyeq .)

There is an extensive literature about the existence of compatible or weakly compatible measures, finitely or σ -additive. In this paper, we shall concentrate on the σ -additive case.

A survey of results, especially for the finitely additive case, appears in [KLST, Chapter 5]. We use the following definition of that book.

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$\langle \mathfrak{B}, \succcurlyeq \rangle$ is a structure of qualitative probability if \mathfrak{B} is a Boolean algebra and \succcurlyeq is a binary relation with field \mathfrak{B} such that:

1. \succcurlyeq is a preorder on \mathfrak{B} , i.e. it is transitive and connected (complete).
2. $0 \neq 1$ and $p \succcurlyeq 0$ for every $p \in \mathfrak{B}$.
3. If $p, q, r, \in \mathfrak{B}$ and $p \cap q = p \cap r = 0$, then $q \succcurlyeq r$ iff $p \cup q \succcurlyeq p \cup r$.

$\langle \mathfrak{B}, \succcurlyeq \rangle$ is a σ -structure of qualitative probability if \mathfrak{B} is a σ -complete Boolean algebra and the following holds in addition to 1, 2 and 3.

4. If $p_i, q \in \mathfrak{B}$ and $p_i \subseteq p_{i+1} \preceq q$ for every $i = 0, 1, 2, \dots$, then $\bigcup_{i < \infty} p_i \preceq q$.
- 3'. $p \succcurlyeq 0$ and $p \neq 0$ for every $p \in \mathfrak{B}$ with $p \neq 0$.

The existence of compatible (or weakly compatible) σ -measures on σ -structures of qualitative probability can be reduced to those that are strictly positive (see [SC, p. 361] for a proof of this fact). So from now on we consider only strictly positive σ -structures of qualitative probability $\langle \mathfrak{B}, \succcurlyeq \rangle$.

Conditions 1–4 are obviously necessary for the existence of a compatible probability measure. Villegas [V] proved that they are also sufficient in case \mathfrak{B} is atomless.

In [SC] it was claimed that they proved this without the assumption that \mathfrak{B} is atomless. However, this theorem is false, as can be easily seen from the example of Kraft, Pratt and Seidenberg [KPS]. This is an example of a finite structure of qualitative probability with no compatible measure (see [KLST, p. 205]).

This example shows that 1–4 are not sufficient in general for the existence of compatible measures. So we add

5. Every finite subalgebra has a compatible measure.

This last condition can be replaced by a purely algebraic condition also due to Kraft, Pratt and Seidenberg [KPS].

5'. For every n and every $p_0, p_1, \dots, p_{n-1} \in \mathfrak{B}$ with $\bigcup_{i < n} p_i = 1$ and $p_i \cap p_j = 0$ for $i \neq j$, if q_0, \dots, q_{m-1} and r_0, \dots, r_{m-1} are in the subalgebra generated by p_0, \dots, p_{n-1} , every p_i is contained in the same number of q_j 's as r_j 's, and $q_j \succcurlyeq r_j$ for $j = 1, \dots, m - 1$, then $r_0 \preceq q_0$.

The purely atomistic infinite case has a mathematical interest in its own right. Since it can be proved that there are countably many atoms, \mathfrak{B} can be viewed as the power set of ω , $\mathbf{P}(\omega)$. Thus, the problem is to find necessary and sufficient conditions on \preceq for the existence of a compatible strictly positive σ -additive probability measure.

We are only able to prove from 1–5 the existence of a weakly compatible strictly positive measure for this case (Theorem 5.3). It remains open whether these conditions are enough for a compatible measure.

On the other hand, when there is a nonempty atomless part, we can prove that 1–5 imply the existence of a compatible measure (Theorem 5.7). Other results of this nature also appear in §§5 and 6.

§2 is devoted to preliminaries and contains definitions and elementary facts that will be used throughout the paper.

§3 is devoted to lemmas, including the fundamental Lemma 3.7, which will be used in proving all the main theorems.

In §4 we consider systems of linear equations and inequalities in nonstandard models of analysis, which, with §3, provide model theoretic proofs of the theorems in §5. At this moment we do not know of any purely algebraic-analytic proof.

Finally, in §7 we present some open problems and discuss the reasons why we consider them important.

2. Preliminaries. Throughout we let $i, j, k, l, m,$ and n denote nonnegative integers. \mathfrak{B} is a σ -Boolean algebra with partial ordering \subseteq . \mathfrak{B} will be identified with its universe and $0, 1, \cup, \cup, \cap, \cap, \sim$ have their usual meanings. p and q will denote arbitrary elements of \mathfrak{B} . We write $p + q$ for $p \cup q$ when $p \cap q = 0$; $\sum p_i$ for $\cup p_i$ when $p_i \cap p_j = 0$ for $i \neq j$.

A preordering on \mathfrak{B} is a transitive, connected binary relation with field \mathfrak{B} , i.e., if $p, q, r \in \mathfrak{B}$ then either $p \leq q$ or $q \leq p$, and if $p \leq q \leq r$ then $p \leq r$. We write $p < q$ if $p \leq q$ but $q \not\leq p$; $p \approx q$ if $p \leq q$ and $q \leq p$.

We introduce binary relation symbols $<$ and \approx (confusing these with their intended interpretations in \mathfrak{B}), regard the elements of \mathfrak{B} as constant symbols, and let Γ be the set of all atomic sentences and negations of atomic sentences in these symbols that are true in \mathfrak{B} .

A probability measure on \mathfrak{B} is a σ -additive nonnegative real valued measure on \mathfrak{B} such that $0 < \mu(1) < \infty$. μ is compatible with \leq if for all $p, q \in \mathfrak{B}$,

$$p \leq q \text{ iff } \mu p \leq \mu q.$$

μ is weakly compatible with \leq if for all $p, q \in \mathfrak{B}$,

$$p \leq q \text{ implies } \mu p \leq \mu q$$

(allowing the possibility that $p < q$ but $\mu p = \mu q$).

Throughout the discussion we assume that \leq has the following properties.

PROPERTY I. If \mathfrak{B}' is a finitely generated subalgebra of \mathfrak{B} then there is a probability measure μ' on \mathfrak{B}' that is compatible with \leq .

PROPERTY II. If $p_0 \subseteq p_1 \subseteq \dots$ and if $p_i \leq q$ for all i , then $\cup p_i \leq q$.

By taking complements it is easy to see that the following is equivalent to II, so we also denote it as

PROPERTY II. If $p_0 \supseteq p_1 \supseteq p_2 \supseteq \dots$ and if $p_i \geq p$ for all i , then $\cap p_i \geq p$.

Note that Properties I and II are necessary for the existence of a probability measure compatible with \leq .

Property I is model theoretic in character but can be replaced by an equivalent set of algebraic conditions as described in §1.

For reasons mentioned in §1, we restrict our attention to atomic \mathfrak{B} except in Theorems 5.7 and 5.8.

The problem of finding a compatible μ for the atomic case reduces to that for \mathfrak{B} atomic and each atom $a > 0$. To see this let \mathcal{I} be the ideal consisting of those $p \approx 0$. Let $\mathfrak{B}^\# = \mathfrak{B}/\mathcal{I}$, the quotient algebra $\mathfrak{B} \text{ mod } \mathcal{I}$. If μ is compatible with \leq/\mathcal{I} we obtain a measure μ compatible with \leq on \mathfrak{B} by defining $\mu p = \mu^*(p/\mathcal{I})$. So from now on we restrict our attention to atomic σ -Boolean algebras \mathfrak{B} in which $a > 0$ for

each atom. Moreover, since Property I gives a compatible measure for \mathfrak{B} finite, we assume that \mathfrak{B} has infinitely many atoms.

We next observe that the atoms of \mathfrak{B} are arranged in a descending sequence. $a_0 \succcurlyeq a_1 \succcurlyeq a_2 \succcurlyeq \dots$ with $a_0 \succ a_{n+1}$ infinitely often. For suppose that we have atoms a_0, a_1, a_2, \dots such that $a_0 \preccurlyeq a_1 \preccurlyeq a_2 \preccurlyeq \dots$. Let $A_n = \sum_{i \geq n} a_i$ for $n = 1, 2, \dots$. Clearly $A_n \supseteq A_{n+1}$ for all n . Moreover, $\bigcap A_n = 0$. Hence by II $a_0 \preccurlyeq 0$, contradicting $a \succ 0$ for all atoms a .

We let $S_n = \sum_{i > n} a_i$.

Unless we state otherwise, we assume all equivalences $p \approx q$ and inequalities $p \succcurlyeq q, p \succ q$ when written are such that $p \subseteq \tilde{q}$.

3. Here we prove several lemmas that will be of use throughout the remainder of the paper. In particular, 3.7 is of central importance.

LEMMA 3.1. *Suppose $p \succ q$. Then there is some n such that $p \cap \tilde{S}_n \succcurlyeq q \cap S_n$.*

PROOF. Suppose $p \preccurlyeq q \cup S_k$ for all k . By Property II, $p \preccurlyeq \bigcap_k (q \cup S_k)$. But $\bigcap_k (q \cup S_k) = q$ so $p \preccurlyeq q$ —contradiction. Hence for some m , $p \succ q \cup S_m$. Now suppose $p \cap \tilde{S}_k \preccurlyeq q \cup S_m$ for all k . Again by Property II, $p = \bigcup_k (p \cap \tilde{S}_k) \preccurlyeq q \cup S_m$ —contradiction. Hence for some l , $p \cap \tilde{S}_l \succ q \cup S_m$. Now take $n = \min\{l, m\}$.

LEMMA 3.2.1. *Suppose $p_0 \prec p_1 \prec p_2 \prec \dots$. Then there is a q such that:*

- (a) *for all k there are infinitely many l such that $q \cap \tilde{S}_k = p_l \cap \tilde{S}_k$; and*
- (b) *$p_l \prec q$ for all l and if $p_l \prec s$ for all l then $q \preccurlyeq s$.*

LEMMA 3.2.2. *Suppose $p_0 \succ p_1 \succ p_2 \succ \dots \succ s \succ 0$. Then there is a q such that:*

- (a) *for all k there are infinitely many l such that $q \cap \tilde{S}_k = p_l \cap \tilde{S}_k$; and*
- (b) *$p_l \succ q$ for all l and if $p_l \succ r$ for all l then $r \preccurlyeq q$.*

PROOF OF 3.2.1. We first claim that there is some $a_n \subseteq p_l$ for infinitely many l ; for if not, then for each n there is a k such that $p_l \subseteq S_n$ for all $l \geq k$. It follows that for each l and for all n , $p_l \preccurlyeq S_n$. Hence by Property II each $p_l \preccurlyeq \bigcap S_n = 0$ and so $p_l \approx 0$, contradicting $p_0 \prec p_1 \prec p_2 \prec \dots$. Hence there is a least n_0 such that $a_{n_0} \subseteq p_l$ for infinitely many l , say $l_0^0, l_1^0, l_2^0, \dots$. Let $p_i^1 = p_{l_i^0}$.

We next claim that there is some m such that $a_{n_0} + a_m \subseteq p_i^1$ for infinitely many i . If not, then for each m there is a k such that $p_i^1 \subseteq a_{n_0} \cup S_n$ for all $l \geq k$ and, hence, as above, $p_i^1 \preccurlyeq \bigcap_n (a_{n_0} \cup S_n)$ for each p_i^1 and so $p_i^1 \preccurlyeq a_{n_0}$. But $p_i^1 \supseteq a_{n_0}$ and so $p_i^1 \approx a_{n_0}$ for all l , contradicting $p_0^1 \prec p_1^1 \prec \dots$. Hence there is a least m , say n_1 , such that $a_{n_0} + a_{n_1} \subseteq p_i^1$ for infinitely many l ; say $l_0^1, l_1^1, l_2^1, \dots$.

Let $p_i^2 = p_{l_i^1}$ and continue.

Take $q = a_{n_0} + a_{n_1} + \dots$.

Clearly q has property (a).

If $q \prec p_m$ for some m then $q \cup S_n \prec p_m$ for some n by 3.1. But then there is a k such that $p_m \prec p_k \subseteq q \cup S_n \prec p_m$ —contradiction. Hence $p_m \preccurlyeq q$ for all m .

Suppose $p_m \preccurlyeq s$ for all m . Then $p_m \cap S_n \prec s$ for all m and n . But for every n there are infinitely many m such that $p_m \cap \tilde{S}_n = q \cap \tilde{S}_n$. Hence $q \cap \tilde{S}_n \preccurlyeq s$ for all n . By Property II, $q \preccurlyeq s$. This proves (b).

PROOF OF 3.2.2. The proof of (a) is similar to that just given for 1 and is left to the reader. To prove (b) suppose that $q \succ p_n$ for some n . By Lemma 3.1 there is a k such that $q \cap \tilde{S}_k \succ p_n$ and so $q \cap \tilde{S}_k \succ p_l$ for all $l \geq k$. But by (a) there is an l such that $q \cap \tilde{S}_k \subseteq p_l$ and $l \geq k$, and so $q \cap \tilde{S}_k \preceq p_l$ —contradiction. Hence $q \preceq p_n$ for all n .

Suppose $r \preceq p_n$ for all n . It follows from (a) that $r \preceq q \cap \tilde{S}_n$ for all n . Hence $r \preceq q$, as needed to complete (b).

LEMMA 3.3. *Suppose $p + p_0 \succ q + q_0$ where $p \cup q \subseteq \tilde{S}_n$ for some n , $p_0, q_0 \subseteq S_n$ and $p + p_1 \approx q + q_1$ for all $p_1, q_1 \subseteq S_n$. Then there is some $p^1 \preceq p_0$ and some $q^1 \succ q_0$, $p^1, q^1 \subseteq S_n$, such that:*

- (i) $p + p_0 \succcurlyeq p + p^1 \succ q + q^1 \succcurlyeq q + q_0$; and
- (ii) if $p + p_0 \succcurlyeq p + p^* \succ q + q^* \succcurlyeq q + q_0$, with $p^*, q^* \subseteq S_n$, then $p^* \succcurlyeq p^1$ and $q^* \preceq q^1$.

PROOF. Let $K = \{q^*: p + p_0 \succ q + q^* \text{ and } S_n \supseteq q^* \supseteq S_m \text{ for some } m\}$. Notice that K is countable and, if $p + p \succ q + q^*$, with $q^\# \subseteq S_n$, then $q^\# \subseteq q^*$ for some $q^* \in K$ by 3.1.

We claim that there is a $q^1 \in K$ such that $q^1 \succcurlyeq q^*$ for all $q^* \in K$. If not then there is an infinite sequence $q_1 < q_2 < q_3 < \dots$ cofinal in K . By Lemma 3.2 there is a $q^\#$ such that $p + p \succ q + q^\#$ and each $q_i < q^\#$. Then by Lemma 3.1 there is a k such that $p + p_0 \succ q + (q^\# \cup S_k)$. We let $q^1 = q^\# \cup S_k$. Clearly q^1 is the maximal member of K .

Now let $L = \{p^*: p + p^* \succ q + q^1, p^* \subseteq S_n \text{ and } p^* \text{ finite}\}$. Clearly L is countable and by Lemma 3.1 if $p + p^\# \succ q + q^1$, with $p^\# \subseteq S_n$, then there is some $p^* \in L$ such that $p^* \subseteq p^\#$.

We claim that L has a minimal member. For if not, then there is an infinite decreasing sequence $p_1 \succ p_2 \succ p_3 \succ \dots$ cofinal in L . If there is no r such that $p_i \succ r$ for all i then clearly for each n there is a k such that for all $l \geq k$, $p_l \subseteq S_m$. So for l large enough, $p_l \subseteq S_m$ and $p + S_m \succ q + q^1$. Hence by Property II we have $p \succ q + q_1$ and so p^1 can be taken as 0, which is in L . If there is some r for which $0 < r < p_i$ for each i , take $p^\#$ to be the q of Lemma 3.2. Since, for each m , $p^\# \cup S_m \supseteq p_i$ for some i we have $p + (p^\# \cup S_m) \succ q + q^1$. Hence, by Property II, $p + p^\# \succ q + q^1$. By 3.1, $p + p^1 \succ q + q^1$ for some finite $p^1 \subseteq p^\#$. But then $p^1 \in L$ and is minimal with respect to ' $<$ '.

It is now easy to see that (i) and (ii) are satisfied.

LEMMA 3.4. *Suppose $p < q$, $p, q \subseteq \tilde{S}_n$ and for no $p^\#, q^\# \subseteq S_n$ do we have $p + p^\# \approx q + q^\#$. Then it is possible to find finitely many pairs $\{\langle p^i, q^i \rangle \text{ for } i \leq n\}$, with $p^i, q^i \subseteq S_n$, such that:*

- (i) $p + p^i \succ q + q^i$ for each i ; and
- (ii) whenever $p + p_0 \succ q + q_0$, $p_0, q_0 \subseteq S_n$, then there is an $i \leq n$ such that $p_0 \succcurlyeq p^i$ and $q_0 \preceq q^i$.

PROOF. Let $K = \{\langle p', q' \rangle: p + p' \succ q + q', \text{ and if } p'' \preceq p' \text{ and } q'' \succcurlyeq q \text{ and } p + p'' \succ q + q'', \text{ then } p'' \approx p' \text{ and } q'' \approx q'\}$. Note that Lemma 3.3 shows that for

any p'' and q'' such that $p + p'' > q + q''$, there is a $\langle p', q' \rangle \in K$ such that $p'' \succcurlyeq p'$ and $q'' \preccurlyeq q'$.

For $\langle p_0, q_0 \rangle, \langle p_1, q_1 \rangle \in K$ define $\langle p_0, q_0 \rangle \sim \langle p_1, q_1 \rangle$ if $p_0 \approx p_1$ and $q_0 \approx q_1$. This defines an equivalence relation on K and we let K' consist of one representative from each equivalence class. Suppose $\langle p_0, q_0 \rangle$ and $\langle p_1, q_1 \rangle$ are distinct members of K' . We cannot have either

(i) $p + p_0 \succcurlyeq p + p_1 > q + q_1 \succcurlyeq q + q_0$ or

(ii) $p + p_0 \succcurlyeq p + p_1 \succcurlyeq q + q_0 \succcurlyeq q + q_1$ since both imply $\langle p_0, q_0 \rangle \sim \langle p_1, q_1 \rangle$.

Hence we must have either

$$p + p_0 > q + q_0 \succcurlyeq p + p_1 > q + q_1$$

or

$$p + p_1 > q + q_1 \succcurlyeq p + p_0 > q + q_0.$$

If the first holds we say that $\langle p_0, q_0 \rangle \succ \langle p_1, q_1 \rangle$. So \succ linearly orders K' . Also, by 3.1 and the definition of K we see that K' is countable.

It will be enough to show that K' is finite.

CLAIM (1). There is no infinite increasing chain in K' of the form

$$\dots \succcurlyeq p + p^1 > q + q^1 \succcurlyeq p + p^0 > q + q^0.$$

Otherwise, by Lemma 3.2, there is a $p^\#$ and a $q^\#$ such that for all n , $p^\# \cap \tilde{S}_n \subseteq p^i$ for infinitely many i and $q^\# \cap \tilde{S}_n \subseteq q^i$ for infinitely many n . Clearly, $p + (p^\# \cup S_m) > q + (q^\# \cap \tilde{S}_n)$ for all m and n . Two applications of Property II give $p + p^\# \succcurlyeq q + q^\#$ and, since $p + p^\# \approx q + q^\#$ is ruled out by hypotheses, $p + p^\# > q + q^\#$. Hence, for some m and n , $p + (p^\# \cap \tilde{S}_m) > p + (q^\# \cup S_m)$. But $q^\# \cup S_m > q^i$ for all i , and for some $\langle p^*, q^* \rangle \in K$ we have $q^* \succcurlyeq q^\#$ —contradiction. This establishes Claim (1).

CLAIM (2). There is no infinite decreasing cofinal chain in K' of the form

$$p + p^0 > q + q^0 \succcurlyeq p + p^1 > q + q^1 \succcurlyeq p + p^2 > q + q^2 \succcurlyeq \dots.$$

If $q^i > r$ for some $r > 0$ and all i , let $q^\#$ be the q of Lemma 3.2. Otherwise let $q^\# = 0$. Similarly, applying 3.2 to $p^0 > p^1 > \dots$ we get $p^\#$. Since for all n there is a k such that $p + (p^\# \cup S_n) > q + q^k > q + q^\#$, it follows by Property II that $p + p^\# > q + q^\#$. Take n such that $p + (p^\# \cap \tilde{S}_n) > q + (q^\# \cup S_n)$ by 3.1. There is an m such that $p^m \supseteq p^\# \cap \tilde{S}_n$ and $q^m < q^\# \cup S_n$. But this is impossible since whenever $p^m \succcurlyeq p''$ and $q^m \preccurlyeq q''$ and $p + p'' > q + q''$, then $p^m \approx p''$ and $q^m \approx q''$. Hence K' has no infinite descending sequence, giving Claim 2.

The two claims now establish the finiteness of K' , completing the proof.

The next lemma has the same form as the preceding except the initial condition $p < q$ is replaced by $p > q$.

LEMMA 3.5. Suppose $p > q$, $p, q \subseteq \tilde{S}_n$ and for no $p^\#, q^\# \subseteq S_n$ do we have $p + p^\# \approx q + q^\#$. Then it is possible to find finitely many pairs $\{\langle p^i, q^i \rangle : i \leq n\}$ such that $p^i, q^i \subseteq S_n$,

(i) $p + p^i > q + q^i$ for each $i \leq n$, and

(ii) if $p + p_0 > q + q_0$, with $p_0, q_0 \subseteq S_n$, then for some $i \leq n$ we have $p_0 \succcurlyeq p^i$ and $q_0 \preccurlyeq q^i$.

PROOF. Let K and K' be as in Lemma 3.4. As before we see that the members of K' form a countable linear ordering given by $\langle p^i, q^i \rangle \succ \langle p^j, q^j \rangle$ iff $p + p^i > q + q^i \succcurlyeq p + p^j > q + q^j$, and the ordering has no infinite increasing subsequence in K' . The proof will be completed by showing that there is no infinite decreasing subsequence in K' .

Using Lemma 3.2 get q^* such that $p > q + q^*$, and whenever $p > q + q^{\#}$, then $q^* \succcurlyeq q^{\#}$. Clearly, $\langle 0, q^* \rangle \in K$ and we can choose K' so as to have $\langle 0, q^* \rangle \in K'$. We cannot have $\langle p_0, q_0 \rangle \in K'$ with $\langle 0, q^* \rangle \succ \langle p_0, q_0 \rangle$, for otherwise $p > p + p_0$. Hence, if there is an infinite decreasing chain it must look like

$$p + p^0 > q + q^0 \succcurlyeq p + p^1 > q + q^1 \succcurlyeq p + p^2 > q + q^2 \succcurlyeq \dots \succcurlyeq p > q + q^*.$$

By Lemma 3.2 we get $q^{\#}$ such that for any n there are infinitely many i such that $q^{\#} \cap \tilde{S}_n = q^i \cap \tilde{S}_n$. (Note that 3.2 yields such a $q^{\#}$ since $q^i > q^*$ for all i .) Again by 3.1 get $p^{\#}$ such that for each n there are infinitely many i such that $p^{\#} \cap \tilde{S}_n = p^i \cap \tilde{S}_n$. Several applications of 3.1 give a k and an l such that

$$p + (p^{\#} \cap \tilde{S}_k) > q + (q^{\#} \cup S_l).$$

For some m , $p^m \supseteq p^{\#} \cap \tilde{S}_k$ and $p^m \neq p^{\#} \cap \tilde{S}_k$ and $q^{\#} \cup S_l \supseteq q^m$ and $q^{\#} \cup S_l \neq q^m$. But then

$$p + p^m > p + (p^{\#} \cap \tilde{S}_k) > q + (q^{\#} \cup S_l) > q + q^m,$$

contradicting $\langle p^m, q^m \rangle \in K$.

DEFINITION 3.6. Let $I_n = \{p: p \subseteq a_0 + a_1 + \dots + a_n\}$ and $T_n = \{p: p \subseteq \sum_{i=n+1}^{\infty} a_i\}$. Let R be the first order theory of the reals with $+$ and $<$. For $\Omega \subseteq \Gamma$ let $\Omega^{\#} = \{p < q: p < q \in \Omega\} \cup \{p = q: p \approx q \in \Omega\}$. Let $\Sigma_n = R \cup \Sigma_n^-$, where Σ_n^- is the set of all expressions of the form $p < q$ or $p = q$, where $p, q \in T_n$, that can be deduced from $R \cup \Gamma^{\#}$.

LEMMA 3.7. For every n there is a finite subset $\Delta_n \subseteq \Gamma$ such that $\Delta_n^{\#} \cup \Sigma_n \vDash \Gamma^{\#}$.

PROOF. We first define Δ_n . Let $p, q \in I_n$.

- (1) If $p < q$ is in Γ put $p < q$ in Δ_n . If $p \approx q$ is in Γ put $p \approx q$ in Δ_n .
- (2) If for some $p_0, q_0 \in T_n$ we have $p + p_0 \approx q + q_0$, put one such instance in Δ_n .
- (3) If for no $p_0, q_0 \in T_n$ do we have $p + p_0 \approx q + q_0$, then put all of the finitely many expressions $p + p^i > q + q^i$ arising in Lemmas 3.4 and 3.5 into Δ_n .

Clearly, any equality $p + p^1 \approx q + q^1$ in Γ is derivable from $\Delta_n \cup \Sigma_n$ by (1) and (2).

Now suppose $p + p_0 > q + q_0$ is in Γ with $p, q \in I_n$ and $p_0, q_0 \in T_n$. Three cases arise.

Case (1). There is some $p^1, q^1 \in T_n$ such that $p + p^1 \approx q + q^1$. In this case there is a $p'' \in T_n$ such that $p + p'' = q + q''$ is in $\Delta_n^{\#}$. In Σ_n we have $q'' + p_0 > p'' + q_0$ (note that the plus signs in the last expression refer to usual addition in R and not to disjoint union in \mathfrak{B}). From this it follows that $\Delta_n^{\#} \cup \Sigma_n \vDash p + p_0 > q + q_0$.

Case (2). $p < q$ and for no $p^1, q^1 \in T_n$ do we have $p^1 + p \approx q^1 + q$. By Lemma 3.4 there is some p^* and q^* such that $p + p_0 \succcurlyeq p + p^* > q + q^* \succcurlyeq q + q_0$, where

$p + p^* > q + q^*$, $p_0 \geq p^*$, $q^* \leq q_0$ are all in $\Delta^* \cup \Sigma_n$. From this it follows that $\Delta_n^* \cup \Sigma_n \models p + p_0 > q + q_0$.

Case (3). $p > q$ and for no $p^1, q^1 \in T_n$ do we have $p^1 + p \approx q^1 + q$. This is handled similarly to (2) by using 3.3.

4. In this section we consider finite systems of inequalities in nonstandard models of analysis.

LEMMA 4.1. *Suppose that \mathfrak{S} is a finite system of equalities of the form*

$$\sum_{j=1}^m \alpha_{ij} x_j = c_j$$

for $j \leq k$, and each α_{ij} is finite and not infinitesimal or zero, and each c_j is finite. If \mathfrak{S} has a nonstandard solution it has a finite solution. (Finite in a nonstandard model means not infinite.)

PROOF. Easy by a Gaussian elimination of variables.

LEMMA 4.2. *Suppose that \mathfrak{S} is a finite system of equalities and inequalities, say $\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2$, where*

$$\mathfrak{S}_1 = \left\{ \sum_{j=1}^m \alpha_{ij} x_j = c_j : j = 1, \dots, k \right\}$$

and

$$\mathfrak{S}_2 = \left\{ \sum_{j=1}^m \beta_{ij} x_j \Delta_j d_j : j = 1, \dots, l \right\}$$

where Δ_j is $<$ or \leq . Also suppose that each α_{ij} and each β_{ij} is finite and not infinitesimal, and each c_j and d_j is finite. If \mathfrak{S} has a solution then \mathfrak{S} has a finite solution.

PROOF. By induction on l . If $l = 0$ the lemma reduces to 4.1. Suppose the statement is true for $l = n$ and let $l = n + 1$. Let C be the solution set for $\mathfrak{S}_1 \cup \mathfrak{S}_2^-$, where $\mathfrak{S}_2^- = \{\sum \beta_{ij} x_j \Delta_j d_j : j = 1, 2, \dots, n\}$. Let σ be the remaining inequality $\sum \beta_{n+1,j} x_j > d_j$ (or $\geq d_j$). Let H be the half-space determined by σ . Three cases arise.

Case (1). $H \cap C = \emptyset$. This is impossible since \mathfrak{S} has a solution.

Case (2). $H \supseteq C$. Then we are done since our induction hypothesis says that C has finite solutions.

Case (3). There are points r and s such that $r \in H \cap C$ and $s \in \bar{H} \cap C$.

Since C is convex, the line segment L joining r to s is contained in C . Hence there is a point t on L lying on the boundary of H . This point satisfies $\mathfrak{S}^\#$ where $\mathfrak{S}^\# = \mathfrak{S}_1 \cup \mathfrak{S}_2^- \cup \{\sigma^\#\}$ and $\sigma^\#$ is $\sum \beta_{n+1,j} x_j = d_j$.

Since $\mathfrak{S}^\#$ has only n inequalities, it has a finite solution $r^\#$. Hence, if σ is a weak inequality we are done.

If σ is a strong inequality we consider the line segment $L^\#$ joining r to $r^\#$. Note that $L^\# \subseteq C \cap \bar{H}$, where \bar{H} is the closure of H .

If r is finite then r satisfies \mathfrak{S} .

If r is infinite then $L^\#$ is infinite in length and the point r^1 of distance 1 from $r^\#$ along $L^\#$ satisfies \mathcal{S} . (Note that r^1 is not on the boundary of H , otherwise $L^\#$ would be contained in the boundary, contradicting the fact that r is not on the boundary.)

5. In this section we prove the existence of probability measures weakly compatible with \leq .

Let $\mathfrak{R} = \langle \mathbf{R}, +, \cdot \rangle$ be the reals with addition and multiplication.

LEMMA 5.1. For each i there is a structure $M_i \equiv \mathfrak{R}$ and a function μ_i on \mathfrak{B} into M_i such that:

- (i) $\mu_i p \leq \mu_i q$ iff $p \leq q$;
- (ii) $\mu_i a_i$ is not infinitesimal;
- (iii) $\mu_i a_0$ is finite and greater than 0.

PROOF. Let $\Sigma = \text{Th } \mathfrak{R} \cup \Gamma^\#$, where $\Gamma^\#$ is as in 3.6. By first order compactness, Σ has a model M .

If a_i^M is not infinitesimal, take $M_i = M$ and for each $p \in \mathfrak{B}$ let $\mu_i p = p^M$.

Suppose that a_i^M is infinitesimal.

We first note that there is some standard natural number n such that $(na_i)^M > S_i^M$. Indeed, it follows from Property II that $a_i > S_j$ for some $j > i$. Hence $S_i = (a_{i+1} + \dots + a_j + S_j) \leq (j - i + 1)a_i$.

Let $\nu p = (p/a_i)^M$. Note that ν satisfies (i) and (ii).

Let \mathcal{S} be the system of equalities and inequalities obtained from Δ_{i-1} by replacing a_j by a variable x_j for $j \leq i - 1$ and each $p \in T_{i-1}$ by νp . \mathcal{S} has a solution in M ; indeed assigning νa_j to x_j gives us a solution. Hence, by Lemma 4.2 there is a finite solution, say the assignment of r_j to x_j . Now let $\mu_i a_j = r_j$ for $j < i$ and $\mu_i a_k = \nu a_k$ for $k \geq i$. If $p = p_0 + p_1$ with $p_0 \in I_i$ and $p_1 \in T_i$, define $\mu_i p = \mu_i p_0 + \nu p_1$, where $\mu_i p_0 = \Sigma\{\mu_i a_k : a_k \subseteq p_0\}$. Clearly this assignment satisfies $\Delta_{i-1}^\# \cup \Sigma_i$ and, hence, by 3.7, $\Gamma^\#$ is satisfied. It follows that (i), (ii) and (iii) are satisfied.

LEMMA 5.2. For each i there is a standard probability measure $\bar{\mu}_i$ on \mathfrak{B} such that

- (i) $\bar{\mu}_i a_i > 0$,
- (ii) $p \geq q$ implies $\bar{\mu}_i p \geq \bar{\mu}_i q$.

PROOF. Let μ_i be as in Lemma 5.1. Let $\bar{\mu}_i p$ be the standard part of $\mu_i p$ for each $p \in \mathfrak{B}$. Clearly (i) and (ii) are satisfied; also $\bar{\mu}_i$ is finite, positive, and finitely additive. It is enough to show that $\bar{\mu}_i$ is monotone. This is clear if $\bar{\mu}_i a_j = 0$ for some j , since then $\bar{\mu}_i a_{j+k} = 0$ for all k and the finite additivity of $\bar{\mu}_i$ gives σ -additivity.

On the other hand, suppose $\bar{\mu}_i a_j > 0$ for all j . Consider a sequence $p_0 \subseteq p_1 \subseteq p_2 \subseteq \dots$ such that $\bar{\mu}_i p_j \leq \bar{\mu}_i q$ for all j .

Case (1). If $\mu_i p_j \leq \mu_i q$ for all j then $p_j \leq q$ for all j , so $\cup_j p_j \leq q$. Hence, $\mu_i(\cup_j p_j) \leq \mu_i q$, so $\bar{\mu}_i(\cup_j p_j) \leq \bar{\mu}_i q$, as needed.

Case (2). Suppose $\mu_i p_j > \mu_i q$ for some j . Then $\mu_i p_j \leq \mu_i p_{j+k}$ for all k , yet $\bar{\mu}_i p_j = \bar{\mu}_i q$, so $\mu_i p_j$ and $\mu_i p_{j+k}$ differ by an infinitesimal for all k . Since $\mu_i a_l$ is not infinitesimal for all l and since μ_i is finitely additive, $p_j = p_{j+k}$ for all k . Hence, $p_j = \cup_t p_t$, so $\bar{\mu}_i \cup_t p_t = \bar{\mu}_i q$.

THEOREM 5.3. *There is a standard probability measure μ on \mathfrak{B} such that:*

- (i) $\mu a_i > 0$ for all i ;
- (ii) $p \succcurlyeq q$ implies $\mu p \geq \mu q$;
- (iii) if $p > q$ and if there are infinitely many n such that $a_n \not\subseteq q$, then $\mu p > \mu q$.

PROOF. Let $\mu p = \sum_{i=0}^{\infty} \bar{\mu}_i(p) \cdot 2^{-(i+1)}$, where $\bar{\mu}_i$ is as in 5.2. Clearly μ satisfies (i) and (ii). Suppose $p > q$ and $a_n \not\subseteq q$ for infinitely many n . By 3.1 there is some k such that $p > q \cup S_k$, and so some a_n such that $p > q + a_n$. By (ii) $\mu p \geq \mu q + \mu a_n$. By (i) $\mu p > \mu q$.

COROLLARY 5.4. Let b_0, b_1, \dots be a subsequence of a_0, a_1, \dots such that $\sum b_i \not\subseteq S_n$ for all n . Let \mathfrak{B}^- be the σ -subalgebra of \mathfrak{B} with elements $\{p: p \subseteq \sum b_i\}$. Then there is a probability measure μ on \mathfrak{B}^- such that for all $p, q \in \mathfrak{B}^-$,

$$p \succcurlyeq q \text{ iff } \mu p \geq \mu q.$$

PROOF. This follows immediately from the previous theorem, noting that (iii) will be satisfied by any $q \in \mathfrak{B}^-$.

COROLLARY 5.5. *There is a (nonstandard) finitely additive measure μ such that*

- (i) $p \succcurlyeq q$ iff $\mu p \geq \mu q$,
- (ii) μa_i is not infinitesimal for any i .

PROOF. The following sentences are consistent with first order analysis.

(1) $f(m, x)$ has domain $N \times \mathfrak{B}$ and range $\subseteq [0, 1]$ (where N is the set of natural numbers).

(2) For each $i \in N$, $f(i, p) \geq f(i, q)$ iff $p \succcurlyeq q$.

(3) $f(j, a_j) > r_j$, where r_j is some standard rational number such that $\mu_j a_j > r_j > 0$ and μ_j is as in Lemma 5.1.

(4) $F(p) = \sum_{j \in N_2} f(j, p)/(j + 2)$.

We consider a model M for these sentences along with a suitable portion of first order analysis and let μp be the value of $F(p)$ in M . Clearly (i) and (ii) are satisfied.

DEFINITION 5.6. Let \mathfrak{B} be a σ Boolean algebra with a countable number of atoms a_0, a_1, a_2, \dots . Let \mathfrak{B}' be the subalgebra generated by the atoms and let \mathfrak{B}'' be the subalgebra consisting of those $b \in \mathfrak{B}$ such that $b \subseteq B''$, where $B'' = \sum a_i$. Let $B^* = \{a_i: a_i \leq B''\}$ and let \mathfrak{B}^* be the smallest σ -algebra containing $B^* = \{p: p \subseteq B''\}$.

THEOREM 5.7. *Let \mathfrak{B} be such that neither \mathfrak{B}' nor \mathfrak{B}'' is empty. Then there is a strictly positive probability measure μ on \mathfrak{B} compatible with \succcurlyeq . Moreover, if ν is another such measure then there is a constant $c > 0$ such that for all $p \in \mathfrak{B}^*$, $\nu p = c\mu p$.*

PROOF. By a lemma in [SC] there is a sequence b_0, b_1, b_2, \dots such that $\mathfrak{B}'' = \sum b_i$ and $b_i \approx \sum_j b_{i+1+j}$ for all i . Let \mathfrak{B}^- be the atomic subalgebra generated by $\{a_0, a_1, \dots\} \cup \{b_0, b_1, \dots\}$. Let Γ^- be the set of all expressions $p \succcurlyeq q$ and $p > q$ in Γ such that $p, q \in \mathfrak{B}^-$. By Theorem 5.5 there is a nonstandard measure ν^- on \mathfrak{B} that is compatible with \succcurlyeq .

Let $\nu^- p = (\nu^- B'')^{-1} \nu p$ for all $p \in \mathfrak{B}^-$.

Of course $\mu^-(B'') = 1$, $\mu^- b_i = 1/2^{i+1}$ and μ^- is compatible with \succcurlyeq .

By Property II there is an $S_m \leq B''$, and hence there is some $a_n \leq B''$.

We claim that $\mu^- a_n$ is standard for all $a_n \leq B''$. To see this it is enough to show that $a_n \approx \sum_{i \in I} b_i$ for some I . Let $b^k = \sum_{i \geq k} b_i$. Since $\bigcap_k b^k = 0$, Property II implies that $a_n > b^k$ for some k and, hence, $a_n > b_l^1$ for some l . Let b_0^1 be the first such. If $a_n \approx b_0^1$, we are done; if $a_n > b_0^1$ there is some b_l such that $a_n \geq b_0^1 + b_l$ (another use of Property II). This gives b_l^1 with $a_n \geq b_0^1 + b_l^1$. Continue the process to get b_0^1, b_1^1, \dots . If the process ends in finitely many steps, we are done. Otherwise we get $a_n \geq \sum b_i^1$. In this case we must get \approx . For if $a_n \not\approx B''$ there is some $b_j \not\subseteq \sum b_i^1$. If there are infinitely many $b_j \not\subseteq \sum b_i^1$, then a slight modification of the proof of 3.1 gives $a_n > \sum b_i^1 + b_j$ for some j , which is impossible by the definition of the b_i^1 's. Otherwise there is a k such that $\sum b_i^1 \approx \sum_{j \in J} b_j^1 + b^{k+1}$, where $b_k \notin \{b_j^1 \mid J \in J\}$. But then $\sum b_i^1 = \sum_{j \in J} b_j^1 + b_k$, again an impossibility by the definition of the b_i^1 's.

Let n be the largest m such that $a_m \leq B''$. Let \mathcal{S} be the system of equalities and inequalities obtained from Δ_n by replacing a_j , with $j \leq n$, by a variable x_j , and each $p \in T_n$ by $\mu^- p$. \mathcal{S} has a solution in a nonstandard model and so has a solution in the reals, say one that assigns r_j to x_j .

Now define $\mu p = \mu^- p$ for all $p \in T_n$ and $\mu \sum^k a_i = \sum^k r_i$ when $\sum^k a_i \in I_n$. The conditions of Lemma 3.7 are met and so μ is compatible with \geq on \mathcal{B}^- .

Let λ be the unique probability measure on \mathcal{B}'' that is compatible with \geq restricted to \mathcal{B}'' . Clearly $\lambda b_i = \mu b_i$ for all i . Now extend μ to all of \mathcal{B} by defining $\mu(p_0 + p_1) = \mu p_0 + \lambda p_1$ when $p_0 \in \mathcal{B}'$ and $p_1 \in \mathcal{B}''$.

We claim that μ is compatible with \geq . For suppose $p = p_0 + p_1$, $q = q_0 + q_1$, $p > q$, $p_0, q_0 \in \mathcal{B}'$ and $p_1, q_1 \in \mathcal{B}''$. Then there is some p_1^1 and q_1^1 such that $p_1^1 \approx p_1$, $q_1^1 \approx q_1$, and $p_1^1 = \sum_{i \in I} b_i$, $q_1^1 = \sum_{j \in J} b_j$ for some I and J . But $p_0 + p_1^1 > q_0 + q_1^1$ and $p_0 + p_1^1, q_0 + q_1^1 \in \mathcal{B}'$, so $\mu(p_0 + p_1^1) > \mu(q_0 + q_1^1)$. A similar argument works for $p \geq q$. Hence, μ is compatible with \geq , as we wanted to show.

For the next theorem, and only for it, notice that the conditions on \mathcal{B} and \geq have been changed.

THEOREM 5.8. *Let \mathcal{B} be a σ -complete Boolean algebra and \geq a binary relation with field \mathcal{B} such that:*

- (i) *Every finite subalgebra admits a probability measure compatible with \geq .*
- (ii) *If for every i , $p_i \subseteq p_{i+1} \leq q$, then $\bigcup_{i < \infty} p_i \leq q$. Under these conditions there is a probability measure μ such that*
 - (a) *If there is a $p \in \mathcal{B}$, such that for every $q \subseteq p$ there is a $q' \subseteq q$ with $0 < q' < q$, then μ is compatible with \geq .*
 - (b) *If for every $p \in \mathcal{B}$ there is a $q \subseteq p$ such that $q' \subseteq q$ implies $q' \sim q$, or $q' \sim 0$, then μ is weakly compatible with \geq .*

PROOF. This theorem is easily reduced to the preceding one by considering the Boolean algebra $\mathcal{B}' = \mathcal{B}/I$, where \mathcal{B}/I is the quotient algebra obtained from \mathcal{B} modulo the σ -ideal of elements equivalent to 0, I .

6. Examples and special cases. We now turn our attention to some additional restrictions on Γ which yield compatible measures, and an example by Steve Leth showing that some Γ force compatible measures to assign irrational numbers to the atoms.

EXAMPLE 6.1 (STEVE LETH). There is a Γ having a unique compatible measure μ such that μa_n is irrational for all n .

Let Γ^- contain the equalities $b_n \approx T_n$ for $n = 0, 1, 2, \dots$, where $b_n = a_{2n} + a_{2n+1}$ and $T_n = \sum_{i=1}^{\infty} b_i$. Clearly, if μ is compatible with Γ^- then $\mu b_n = 1/2^{n+1}$. For each n choose an irrational $c_n \in (1/2^{n+2} - 1/2^{n+4}, 1/2^{n+2})$ and let $d_n = 1/2^{n+1} - c_n$. Clearly, $c_0 > d_0 > c_1 > d_1 > c_2 > d_2 > \dots$. Each c can be expressed as a sum $\sum\{2^{-i}: i \in I_n\}$, and each d_n can be expressed as a sum $\sum\{2^{-j}: j \in J_n\}$. We now add to Γ^- all equalities of the form $a_{2n} \approx \sum\{a_i: i \in I_n\}$ and $a_{2n+1} \approx \sum\{a_j: j \in J_n\}$. Clearly, there is a unique measure μ compatible with Γ^- , and $\mu a_{2n} = c_n$ and $\mu a_{2n+1} = d_n$ for all n . Now Γ^- is completed to a total preordering Γ in the obvious way.

THEOREM 6.2. *If $a_n \leq S_n$ for all n then there is a probability measure compatible with \succcurlyeq .*

PROOF OF 6.2. We need several lemmas.

LEMMA 6.3. *For every atom a_k there is a $q \subseteq S_k$ such that $a_k \approx q$.*

PROOF. Let $a_{n_0} = a_{k+1}$ and, if $a_k \succ \sum^s a_{n_i}$, let $a_{n_{i+1}}$ be the atom of lowest index such that $a_k \succ \sum^{s+1} a_{n_i}$. Let $q = \sum^\infty a_{n_i}$. We claim that $q \approx a_k$. Surely $q \leq a_k$ by monotonicity. If $q < a_k$ then $q \supseteq S_n$ for some n by Lemma 3.1 and the definition of q . Let n^* be the least n such that $q \supseteq S_n$. Note $n^* \neq k$ since then $q \approx S_k$, but $a_k \leq S_k$. But if $n^* > k$ then $a_{n^*} \not\subseteq q$ and $a_{n^*} \leq S_{n^*}$, contrary to the choice of the a_{n_i} . Hence, $q < a_k$ is impossible, and we have $q \approx a_k$.

LEMMA 6.4. *For every a_k there is a $q \subseteq S_k$ such that $a_k \approx q$ and $q \supseteq a_l + a_m$ for some l and m .*

PROOF. There are only finitely many i such that $a_k \approx a_i$ (as shown in §2). Now apply 6.3 to the atom a_{n_i} of highest index.

LEMMA 6.5. *For every p there is some q such that $p \approx q$ and $q \supseteq a_n$ for infinitely many n .*

PROOF. If $p = p^1 + a_{k_0}$, where $p^1 \subseteq a_0 + a_1 + \dots + a_{k_0-1}$, we apply 6.4 to a_{k_0} getting $p \approx p^1 + q_0$. If $q_0 = q_0^1 + a_{k_1}$, where $q_0^1 \subseteq a_0 + a_1 + \dots + a_{k_1-1}$, we apply 6.4 to a_{k_1} , getting $p = p^1 + q_0^1 + q_1$. We continue the procedure getting q_1, q_2, \dots and q_1^1, q_2^1, \dots such that $p = p^1 + q_0^1 + q_1^1 + \dots + q_n$. If some $q_n \supseteq a_l$ for infinitely many l , we are done. If not, we let $q = p^1 + q_0^1 + q_1^1 + \dots$. We claim that $p \approx q$. Clearly,

$$q \cup S_n \supseteq p^1 + q_0^1 + q_1^1 + \dots + q_l \approx p$$

for all l and n , so by Property II, $q \succcurlyeq p$. On the other hand, for all n there is an l such that $q \cap \tilde{S}_n \subseteq p^1 + q_0^1 + q_1^1 + \dots + q_l \approx p$, so by Property II, $q \leq p$. It follows that $q \approx p$ as needed.

For the existence of a weakly compatible measure μ we apply 5.3. In fact μ is a strongly compatible measure. For suppose $p \succ r$. By 5.3(iii) if $r \not\supseteq S_n$ for all n , then $\mu p > \mu r$. On the other hand, if $r \supseteq S_n$ for some n , we use 6.5 to get $q \approx p$ with

$q \supseteq a_k$ for infinitely many k . There are q_1, q^*, r_1 such that $q = q_1 + q^*$ and $r = r_1 + q^*$. By 5.3(iii) $\mu q_1 > \mu r_1$, from which it follows that $\mu q > \mu r$ and $\mu p > \mu r$. This shows that μ is compatible with Γ .

We suspect that the condition $a_n \preceq S_n$ for all n implies the uniqueness of the compatible measure, but we do not see how to prove or disprove this.

THEOREM 6.6. *Suppose that $a_n \succcurlyeq S_n$ for all n . Let $\sum r_i$ be a series of real numbers such that:*

- (i) $\sum r_i = 1$;
- (ii) $r_n = \sum_{i=n} r_{i+1}$ whenever $a_n \approx S_n$;
- (iii) $r_n > \sum_{i=n} r_{i+1}$ whenever $a_n > S_n$.

Let $\mu p = \sum \{r_i: a_i \subseteq p\}$. Then μ is a probability measure compatible with \succcurlyeq . In particular, if $a_n \succcurlyeq S_n$ for all n and $a_k > S_k$ for at least one k , then there are infinitely many probability measures compatible with \succcurlyeq .

PROOF. Let μ be as above and suppose $p > q$. Without loss of generality we can suppose $p \subseteq \tilde{q}$. Let m be the least n such that $a_n \subseteq p$ or $a_n \subseteq q$. Since $a_m \succcurlyeq S_m$ and either $p \subseteq S_m$ or $q \subseteq S_m$, we must have $a_m \subseteq p$. Then $\mu p \geq \mu a_m \geq \mu S_m \geq \mu q$. If $\mu p = \mu q$ then $q = S_m$ and $p = a_m$ (by the elementary properties of the series $\sum r_i$), so $a_m \approx S_m$ by (ii) and (iii)—contradiction. Hence $\mu p > \mu q$.

An easy modification of the argument just given shows that if $p \approx q$ then $\mu p = \mu q$.

At this time we are unable to prove that if $a_n \succcurlyeq S_n$ for all n greater than some n^* then \succcurlyeq has a compatible probability measure.

7. Problems and remarks. We conclude with some open questions and a few remarks concerning our interest in them.

Are the conditions on Γ (i.e., finite satisfiability and monotonicity) sufficient for the existence of a compatible measure? If not, then a counterexample might lead to additional conditions, algebraic or analytic in flavor, that are sufficient.

This paper considers only total (i.e., complete) preorderings, where every pair of elements $p, q \in \mathfrak{B}$ are comparable, $p \preceq q$ or $q \preceq p$. There are several reasons why it is desirable to extend these results to partial preorderings where not every pair p, q can be compared. Intuitively, we have no way of describing all of the uncountably many events (subsets of \mathfrak{B}), never mind comparing them. Moreover, in many concrete situations our interest is only in a finite or countably infinite set of events, and a measure compatible with these events would be sufficient. Secondly, generalizations of this kind open the way to mathematical applications in which different measures have to be extended or combined.

A further step in generalizing the results from total preorderings to partial preorderings is to demand of some of the events that they be assigned particular numerical values. Intuitively, some of the events might be simple enough to determine a probability while others might be so complicated that only comparative probabilities can be given initially. Thus a compatible measure would assign to some of the events their predetermined value and would honor all the comparisons in Γ as well.

There are some cases, for example if $a_n \approx S_n$ for all n , when Γ admits only one compatible measure. There are other cases, for example when $a_n > S_n$ for all n , when Γ admits many compatible measures. Can we characterize those Γ that admit a unique compatible measure, or at least can we find very general conditions which assure a unique compatible measure?

Let \mathfrak{B}^- be the subalgebra of the atomic Boolean algebra \mathfrak{B} generated by the *finite* Boolean operations from the atoms of \mathfrak{B} , a_0, a_1, a_2, \dots . Let Γ^- be the set of equalities and inequalities of Γ which mention only the elements of \mathfrak{B}^- . If μ^- is a measure on \mathfrak{B}^- compatible with Γ^- then μ^- is uniquely extendable to a probability measure μ on \mathfrak{B} honoring Γ (an application of the Carathéodory Extension Theorem). *Notice that Γ^- is countable*, so we can talk about a recursive or recursively enumerable Γ^- . Several problems now arise naturally. If Γ^- is recursive or recursively enumerable and a compatible measure exists, is there a recursive compatible measure? (Here we want a recursive function $f(n, m)$ such that $f(n, m)$ is the m th decimal place in μa_n , or a function $f(n, m)$ such that when $n = p_{i_1} p_{i_2} \cdots p_{i_k}$, where the p 's are distinct primes, then $f(n, m)$ is the m th decimal place in $a_{i_1} + a_{i_2} + \cdots + a_{i_k}$.)

ADDED IN PROOF. Steve Leth has recently proved that the condition $a_n \leq S_n$ for all n implies that the compatible measure of Theorem 6.2 is unique, answering the question posed after the proof of 6.2. He also constructed a counterexample to the third problem of §7 in which one term is assigned a fixed value and no compatible measure exists.

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